

Special Points of Inflation in Flux Compactifications

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ABSTRACT: We study the realisation of axion inflation models in the complex structure moduli spaces of Calabi-Yau threefolds and fourfolds. The axions arise close to special points of these moduli spaces that admit discrete monodromy symmetries of infinite order. Examples include the large complex structure point and conifold point, but can be of more general nature. In Type IIB and F-theory compactifications the geometric axions receive a scalar potential from a flux-induced superpotential. We find toy variants of various inflationary potentials including the ones for natural inflation of one or multiple axions, or axion monodromy inflation with polynomial potential. Interesting examples are also given by mirror geometries of torus fibrations with Mordell-Weil group of rank $N - 1$ or an N -section, which admit an axion if $N > 3$.

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1 Introduction

The realization of inflationary models in string theory is a long-standing challenge [1]. A large class of models of inflation employ the dynamics of one or more scalar field that slowly roll down a flat potential. While there has been progress in understanding candidate scalar potentials that arise in string theory [2, 3], it remains challenging to identify scalars with sufficiently flat potentials that at the same time are the lightest scalar degrees of freedom during the inflationary epoch. In large field inflationary models this task becomes even more demanding, since the flatness of the potential and stability of other field space direction has to be controlled over super-Planckian distances. Large field inflationary models that predict a large tensor-to-scalar ratio have recently gained much attention due to the initial claim of the BICEP2 experiment to having discovered primordial gravitation waves [4]. While this result is still under debate [5], it is in any case an interesting conceptual task to realize large field inflation in string theory.

Large field inflationary models can be constructed, for example, by considering scalars that have an axion-like shift-symmetry in the absence of a scalar potential. This symmetry is then broken by a scalar potential, which has to be controlled over a large field range. Candidates for such axions are zero modes of the R-R and NS-NS form fields of string theory. A scalar potential for these fields can arise from brane or flux backgrounds or can be induced non-perturbatively by brane-instantons or gaugino condensates. Prominent scenarios implementing these steps are aligned axion inflation models [6–9], models of N-flation [10–14], and models of axion monodromy inflation [15–26]. While the necessary ingredients for these models are present in string theory, the explicit realization of inflation in combination with moduli stabilization is challenging. The situation improves if the effective theory implementing the model preserves $\mathcal{N} = 1$ supersymmetry with a scalar sector described by a Kähler potential and superpotential. For example, models of axion monodromy inflation were suggested to also arise in supersymmetric theories from an F-term breaking [19, 21]. For all of these scenarios, it should be stressed that the shift symmetries of the fields for the kinetic terms are only present at special points in moduli space, at which the extended objects coupling to the form-fields are sufficiently heavy.

In this paper we study a rich class of axion inflationary models arising from the axions being realized in the complex structure moduli space of the internal manifold. We consider mainly Type IIB orientifold compactifications based on Calabi-Yau threefolds [2, 3, 27], but also comment on the generalization to F-theory on elliptically fibered Calabi-Yau fourfolds [28, 29]. For such Calabi-Yau geometries we suggest that one

can systematically identify special points in their complex structure moduli space at which scalars exist that have approximately shift-symmetric kinetic terms. In fact, these special points are often connected by dualities to string theory setups in which the axions are R-R or NS-NS form-field zero modes. One example, which has recently been investigated in this context [20, 30–32] is the ‘large complex structure point’ in the moduli space of the Calabi-Yau geometries. For Calabi-Yau threefold compactifications of the Type II string theories one can use mirror symmetry to identify the large complex structure point with the large volume point of a dual geometry. In this situation the classical shift symmetry of the NS-NS two-form is the dual source of the shift symmetry. With this understanding one can also gain more control over the expected sources that break the shift-symmetry. For example, at the large volume point of Calabi-Yau threefolds one can easily see that the continuous symmetry will be broken by world-sheet instantons.

In order to identify points in the complex structure moduli space at which axions exist, we propose the following strategy: the complex structure of a Calabi-Yau n -fold can be parameterized by the integrals of the holomorphic $(n, 0)$ form over a integral symplectic basis of $H_n(X_n, \mathbb{Z})$. These integrals are known as the *periods* of X_n . At certain special points in complex structure moduli space these periods can have orbifold or log singularities. One can then encircle these points and study the behavior of the periods, which are typically not single-valued but rather admit *discrete monodromies* when encircling these points. These discrete symmetries, however, do in general not suffice to infer the presence of axions with *approximate continuous shift-symmetries*. We therefore have to further constrain our considerations to special points in complex structure moduli space for which the local monodromy transformation ensures that indeed the Kähler potential admits a shift symmetry. We will present some conditions on the local monodromy that allows for approximate shift symmetries to exist.

A scalar potential for the complex structure moduli in Type IIB Calabi-Yau orientifold and F-theory compactifications is induced upon switching on background fluxes [2, 3]. More precisely, one can show that this potential arises from a superpotential depending on the values of the periods [33, 34]. For sufficiently generic fluxes the superpotential breaks the discrete monodromy symmetry as well as the local continuous shift symmetry. Evaluating this superpotential at different special points that admit axions, we show that different types of shapes of scalar potentials are induced. For example, while the potential is of polynomial-type near the large complex structure point, it is of cosine-type near a conifold point. Therefore a number of different inflation models arise naturally from the effective theories that we get when we move close to different special points in the moduli space.

To illustrate our proposal we will investigate several Calabi-Yau threefold examples in detail and compute the explicit moduli dependence of the periods at special points of the moduli space. A first class of examples are one-parameter Calabi-Yau manifolds. These have been investigated in the study of mirror symmetry before in [35–42]. A second class of examples are Calabi-Yau threefolds that are mirror dual to elliptic fibrations. We show that these admit special points admitting axions if the elliptic fibration has either Mordell-Weil group of rank $N - 1$ or an N -section with $N > 3$. Near a certain ‘small complex structure point’ we identify the axion in the resulting models of *Mordell-Weil inflation*.

Before starting our investigations let us stress that our aim is not to construct completely realistic inflation models, since this would require a detailed study of moduli stabilization. We rather hope to argue that the complex structure moduli space of Calabi-Yau manifolds is rich enough to offer all the necessary building blocks to engineer a large class of phenomenologically appealing axion inflation models. It also allows to compute corrections and study stability of the vacua.

The paper is organized as follows. In section 2 we first briefly introduce Type IIB orientifold setups and their F-theory generalizations. We discuss the form of the Kähler potential and flux superpotential with particular focus on the complex structure moduli dependence. This will allow us to describe the basic idea to identify axions in this moduli space. We will also give already a brief summary of the results with detailed computations carried out in sections 3 and 4. In section 3 we study periods of one-parameter Calabi-Yau threefolds in detail and provide the computation of the Kähler potential and flux superpotential at various special points in moduli space. In the final section 4 we turn to the study of the complex structure moduli space for mirror threefolds of elliptic fibrations.

2 Inflation at special points in complex structure moduli space

In this section we introduce our setup and describe the key steps to identify axions in complex structure moduli space. The Type IIB orientifold compactifications on Calabi-Yau threefolds will be discussed in subsection 2.1. We introduce the general form of the Kähler potential and flux superpotential with focus on the complex structure moduli. A discussion of the monodromy group generated due to special points in moduli space allows us to state the geometric requires for the existence of axions. Some of our main results on the resulting form of the Kähler potentials and flux superpotentials at certain special points in moduli space are described in subsection 2.2. Finally, we comment on

the generalization to F-theory compactifications on fourfolds in subsection 2.3.

2.1 Axions at special points in orientifold set-ups

Most of our discussion will take place in the context of Type IIB Calabi-Yau orientifold compactifications with $O7^-$ planes. In order to cancel tadpoles these set-ups will also include space-time filling D7-branes and fluxes. The resulting system can preserve (possibly spontaneously broken) $\mathcal{N} = 1$ supersymmetry, we will be focusing on situations in which this is the case. Such compactifications allow many appealing features from the point of view of string model building [2, 3]. They also arise as the weak string coupling limit of F-theory compactifications as we will briefly discuss in section 2.3.

Before recalling aspects of the $\mathcal{N} = 1$ effective theory relevant to this work, let us quickly summarize some facts about the construction of orientifold models following [34, 43]. The starting point is a Calabi-Yau threefold X_3 admitting a holomorphic \mathbb{Z}_2 involution $\sigma: X_3 \rightarrow X_3$. We demand that σ acts on the holomorphic $(3, 0)$ -form Ω on X_3 as

$$\sigma^* \Omega = -\Omega . \quad (2.1)$$

The physical model is constructed by quotienting Type IIB string theory on X_3 by $(-1)^{F_L} \sigma$, with Ω_p the orientation reversal action on the worldsheet, and F_L the left-moving fermion number. The fixed loci of such an involution are divisors of X_3 , which we identify with $O7$ planes, and possibly points in X_3 , which get identified with $O3$ planes.

The general four-dimensional $\mathcal{N} = 1$ effective theory for the bulk moduli of such orientifold set-ups has been worked out in [43]. Supersymmetry implies that the dynamics of the fields can be encoded by a Kähler potential K and a superpotential W . In the following we will focus on the moduli sector of such theories. More precisely, we will denote by $\tau = C_0 + ie^{-\phi}$ the dilaton-axion field, and by z^k the $h_-^{2,1}(X_3)$ complex structure moduli compatible with (2.1). The $(3, 0)$ -form Ω depends on the complex structure moduli z^k . At classical order, the Kähler potential for τ, z^k takes the form

$$K = -\log [i(\tau - \bar{\tau})] - \log \left[i \int_{X_3} \Omega \wedge \bar{\Omega} \right] + \dots \quad (2.2)$$

where the dots indicate terms depending on other moduli or matter fields of the set-up. A superpotential is induced by R-R and NS-NS background fluxes F_3, H_3 [33, 34]. Defining $G_3 = F_3 - \tau H_3$ it takes the form

$$W = \int_{X_3} G_3 \wedge \Omega + \dots \quad (2.3)$$

where the dots denote non-perturbative corrections that we will not discuss in detail in this work. Let us stress that in principle one has to control all other terms in (2.2) and (2.3) in building an inflationary model with complete moduli stabilization. This is expected to be challenging, since the z^k, τ are known to mix with other fields in the suppressed terms in (2.2) and (2.3). Recent more complete studies of moduli stabilization in related settings can be found in [30–32]. To convey our message about the existence of axions and the varying shapes of the scalar potential we will assume that we can study the dynamics of the z^k using the displayed terms in (2.2) and (2.3). Furthermore, we will consider the case $h_+^{2,1}(Y_4) = 0$, such that the orientifold projection does not complicate our study additionally. This condition can be weakened, the generic situation can be naturally studied in the F-theory setting discussed in section 2.3.

We study in this work inflationary models for which the inflaton is among the complex structure moduli z^k . It will be therefore crucial to examine the dependence of Ω on z^k . In order to do that it is convenient to choose a basis \mathcal{A}_i for $H_3(X_3, \mathbb{Z})$ and define the periods

$$\Pi^i = \int_{\mathcal{A}_i} \Omega , \quad (2.4)$$

Clearly, the Π^i are depending on the fields z^k through Ω . Furthermore, the periods transform under $Sp(2(h^{2,1} + 1))$, which gives the freedom to choose a symplectic basis \mathcal{A}_i with

$$\mathcal{A}_i \cap \mathcal{A}_j = \eta_{ij} , \quad \eta_{ij} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} , \quad (2.5)$$

where $\mathbf{1}$ is the $(h^{2,1} + 1) \times (h^{2,1} + 1)$ unit matrix. The action of the group $Sp(2(h^{2,1} + 1))$ on the periods Π^i is via

$$\Pi'^i = T^i_j \Pi^j . \quad (2.6)$$

keeping the Kähler potential invariant. We stress though that (2.6) is in general not a symmetry of the system. There is, however, a concept of symmetry group $G_{\text{mon}} \subset Sp(2(h^{2,1} + 1))$, known as the monodromy group, which we will introduce below. To evaluate its action on the physical quantities we note that K and W given in (2.2), (2.3) can be written as

$$K = -\log [i(\tau - \bar{\tau})] - \log [i\Pi^i \eta_{ij} \bar{\Pi}^j] + \dots \quad (2.7)$$

$$W = \mathcal{N}_i \Pi^i + \dots , \quad \mathcal{N}_i = N_i - \tau M_i , \quad (2.8)$$

where $N^i = \int_{\mathcal{A}_i} F_3$ and $M^i = \int_{\mathcal{A}_i} H_3$ are the flux quanta.

A special role will be played by the discrete group G_{mon} . This group is an actual symmetry of the Kähler potential terms displayed in (2.7) independent of the considered

point in complex structure moduli space. To introduce this group we first note that the moduli space of complex structure generally admits special points that are given by the loci at which some of the periods Π^i become singular. If one marks these points in the moduli space one can now encircle them and investigate how the periods transform. Let us consider one such a point z_s and denote the matrix providing a symmetry transformation of the periods by $T_i^j[z_s]$, i.e.

$$T_i^j[z_s] \Pi^i(z) = \Pi^j(z) . \quad (2.9)$$

Collecting all such $T_i^j[z_s]$ for all special points, one can show that they form a group $G_{\text{mon}} \subset Sp(2(h^{2,1} + 1))$. This monodromy group and special points will play the key role in this work.

To be more precise, our interest in this paper is on identifying candidates for *large-field* inflationary models that arise close to the special points z_s in complex structure moduli space. Our strategy is given by

- We aim to identify points in the moduli space near which the Kähler potential K given in (2.2) has an approximate continuous shift symmetry, coming from going around the marked point in moduli space. So we have a natural candidate for an *axion* θ , which we will be able to identify in specific examples. We claim that this arises, in particular, at special points z_s for which the monodromy group element $T_i^j[z_s] \in G_{\text{mon}}$ acting on the periods is of *infinite order*, i.e. there exists no n such that $T[z_s]^n = T[z_s]$.¹ The leading order behavior of K can then be computed using techniques for period computations [35–42] or the recent direct approach using localization [44–49].
- In a second step we then consider the flux induced superpotential W given in (2.3). If the vectors M^i, N^i of chosen fluxes are not invariant under $T_i^j[z_s]$ the superpotential breaks the approximate shift symmetry of the Kähler potential spontaneously. In other words, there is a flux induced scalar potential for the axion θ . The fact, that we restricted our considerations to monodromies of infinite order implies that in principle one can keep increasing the energy of the inflaton θ indefinitely. The two situations of finite and infinite order are depicted in Figure 1.

¹Let us stress that only in the case of infinite order monodromies we are able to directly identify an axion. Nevertheless it would be interesting to investigate finite order situations with finite order monodromy from a phenomenological point of view.

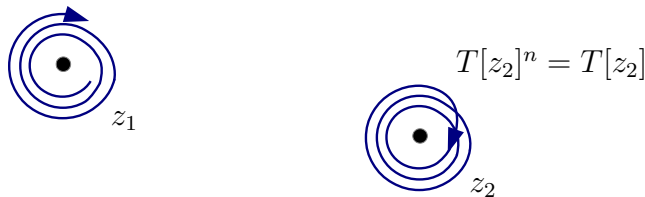


Figure 1. Schematic depiction of singular points in complex structure moduli space with infinite-order monodromy z_1 and finite-order monodromy z_2 . An axions can be identified very close to the point z_1 .

The main aim of this paper is to show that there are indeed rather simple situations in which such special inflationary points in moduli space arise. We will discuss two main families of examples in the later sections: (1) one-parameter Calabi-Yau models (i.e. Calabi-Yau threefolds with a single complex structure deformation), and (2) Calabi-Yau threefolds with the mirror geometry being torus fibered, in which the problem, taking a limit in the parameters describing the base, reduces effectively again to a one-parameter model. We will see that in both cases there are interesting special points, beyond the large complex structure limit previously studied in detail [20, 30–32].

2.2 Summary of results for Calabi-Yau threefolds

While presenting the details of the individual models will require to introduce more mathematics, as done in sections 3 and 4, it is instructive to already have a first look at the results. In particular, one indeed finds that very different inflationary potentials arise at different points in moduli space. Examples are the ‘large complex structure point’, the ‘small complex structure point’ and the ‘conifold point’. We will discuss the various results on these points in the following

- *Large complex structure point:* The analysis of the large complex structure point is related by mirror symmetry to the large volume behavior of a mirror-dual Calabi-Yau threefold Y_3 to our original manifold X_3 . We will stepwise introduce the systematics for computing the periods expanded around the large complex structure point in section 3 and show that the monodromy is of infinite order. Restricting to one-parameter models, one indeed finds an axion that upon choosing appropriate coordinates can be identified with the real part of a complex scalar t . The large complex structure point is located at $t = i\infty$. The Kähler potential K and flux superpotential W expanded around this point in moduli space take

the form

$$K_{\text{lcs}} = -\log [i(\tau - \bar{\tau})] - \log \left[i \left(-\frac{1}{6} \mathcal{K}(t - \bar{t})^3 + 2\hat{c} \right) \right] , \quad (2.10)$$

$$W_{\text{lcs}} = \frac{1}{6} \mathcal{N}_4 \mathcal{K} t^3 - \frac{1}{2} \mathcal{N}_3 \mathcal{K} t^2 + \left(\mathcal{N}_4 \hat{b} + \frac{1}{2} \mathcal{N}_3 \mathcal{K} + \mathcal{N}_2 \right) t + \left(\mathcal{N}_1 - \mathcal{N}_4 \hat{c} + \mathcal{N}_3 \hat{b} \right) ,$$

where \mathcal{N}_i was defined in (2.8) and $\hat{c} = \frac{\zeta(3)\chi}{(2\pi i)^3}$. The topological numbers \mathcal{K} , \hat{b} , and χ of the mirror geometry Y_3 to X_3 will be introduced in (3.13). Let us note that (2.10) yields the polynomial-type potentials that have been discussed in more detail in [19, 20, 30–32].

- *Small complex structure point:* A second special point in one-parameter models is known as the small complex structure point. Upon choosing an appropriate complex coordinate u , with the small complex structure point located at $|u| = \infty$, we aim to identify the axion with the phase of the complex coordinate u . It should be stressed, however, that at this special point the phase has not necessarily an approximate shift-symmetry for each geometry. Concretely, we consider a geometry X_3 with mirror geometry being the complete intersection $Y_3 = \mathbb{P}^n[d_1 \dots d_k]$, meaning the complete intersection of the hypersurfaces of degree d_1, \dots, d_k in \mathbb{P}^n . This allows to define the set $\{\alpha_i\} = \{1/d_1, \dots, (d_1 - 1)/d_1, 1/d_2, \dots, (d_k - 1)/d_k\}$. One now checks that only if at least two α_i coincide, one actually finds a monodromy matrix of infinite order and therefore an axion. In this case, the leading Kähler potential and superpotential expanded around such points in moduli space take the schematic form

$$K_{\text{scs}} = -\log [i(\tau - \bar{\tau})] - \log [a|u|^{-2\alpha_\kappa} \log |u| + \dots] , \quad (2.11)$$

$$W_{\text{scs}} = \sum_i N_i^{\text{eff}} u^{-\alpha_i} (\log(u) + \dots) + \sum_j N_j^{\text{eff}} u^{-\alpha_j} + \dots$$

This expansion displays only the leading terms. In particular, α_κ is the smallest repeated constant in the set $\{\alpha_i\}$. The first sum in W_{scs} is running over the repeated α_i and the second sum over the non-repeated ones, as we will explain in more detail in section 3. The parameter a is a complex function of the α_i and the topological numbers of Y_3 , while the N_i^{eff} , N_j^{eff} are linear combinations of the fluxes $\mathcal{N}_1, \dots, \mathcal{N}_4$ with coefficients dependent on the α_i and topological numbers Y_3 . (We will fully determine these quantities in our examples in section 3.)

- *Conifold point:* A third special point of interest is the so-called conifold point. In a one-parameter model one can choose a coordinate t_c such that it is located at

$|t_c| = 0$, with the axion being the phase θ of t_c . Expanding the Kähler potential and superpotential around this point one finds

$$K_{\text{con}} = -\log [i(\tau - \bar{\tau})] - \log \left[\frac{-1}{\pi} \mathcal{K} |t_c|^2 \log |t_c| + \dots \right] , \quad (2.12)$$

$$W_{\text{con}} = -\mathcal{N}_4 \mathcal{K} t_c \log(t_c) + \frac{1}{2} \mathcal{N}_3 \mathcal{K} t_c^2 + \left(\mathcal{N}_2 - \frac{1}{2} \mathcal{N}_4 \mathcal{K} \right) t_c + \mathcal{N}_1 .$$

It is interesting to point out that the scalar potential derived from K_{con} , W_{con} admits $\cos(\theta)$ as a leading term, i.e. a periodic potential used in models of natural inflation [50, 51]. This is not unexpected, since, at least in geometries with more than one complex structure modulus, one can sometimes perform a geometric transition replacing the fluxes with a stack of five-branes [52–56]. These branes induce a non-perturbative superpotential due to a gaugino condensate resulting in a $\cos(\theta)$ term in the scalar potential.

While a detailed phenomenological analysis of the various occurring potentials is beyond the scope of this paper, it is intriguing to realize that already the simple examples above yield different inflationary potentials.

Let us close this section by also commenting on some generic special points that arises for Calabi-Yau threefolds X_3 for which the mirror geometry Y_3 is torus-fibered. Such torus-fibered Y_3 admit generically a new type of special locus in their Kähler moduli space at which the corresponding geometries Y_3 admit a torus fiber with volume shrunken to zero. In the mirror X_3 this corresponds to a special locus in the complex structure moduli space. While the general analysis of the Kähler potential and superpotential near this locus is rather involved, one can further consider the limit in which the base of Y_3 is large, i.e. Y_3 , X_3 behave in the base direction as in the discussion of the ‘large complex structure point’ above. This mixed limit yields a special point in the complex structure moduli space of X_3 that we term the ‘F-point’.²

- *F-point*: Studying the monodromy around the F-point, we will see in section 4 that the existence of axions at this point depends on the structure of sections of Y_3 . More precisely, if Y_3 admits an elliptic fibration with N sections, i.e. the Mordell-Weil group has rank $N - 1$, then an axion exists for $N > 3$. Alternatively, if Y_3 has an N -section the same bound $N > 3$ applies for having an axion. The Kähler potential and superpotential for the mirrors of elliptic fibrations can be

²We note that this limit coincides on the Kähler moduli side with the F-theory limit of an M-theory setup [29, 57].

evaluated at various points in moduli space similar to the approach of section 3.³ Remarkably, when $N > 3$ the periods near the F-point admit a new symmetry that allows to map their structure to the large complex structure point for both base and fiber. Therefore, one expects that the leading terms of K, W will be of the type (2.10), but generalized to include more than one modulus. One furthermore finds that the coefficients of the various terms depend on the topological data of the torus fibered Y_3 in a distinguished way and will single out the axion associated to the torus fiber.

We will call the models arising near the F-point, *Mordell-Weil inflation*. While the physical setup is somewhat different to ours, the mathematical techniques recently developed for the study of F-theory compactifications with extra $U(1)$ symmetries [60–78] will be directly applicable to determine the topological data required for the construction of inflationary models suggested here. It is an exciting task to carry out this analysis further and check if the constraints on moduli stabilization recently found in [30–32] can be met in these setups.

Let us close this section by noting that the functional forms for K, W encountered in (2.10), (2.11), and (2.12) are not the only possibilities of situations that might occur, but rather are the results of our explicit analysis in sections 3 and 4. It is an interesting task to extend the list of special points and study other candidate Kähler potentials and superpotentials. This becomes particularly interesting when extending the analysis to Calabi-Yau fourfolds use in the F-theory generalization discussed next.

2.3 Generalization to F-theory

In this section we discuss the generalization of the orientifold construction of section 2.1 to F-theory. The F-theory set-up of interest are compactifications on Calabi-Yau fourfolds X_4 that are elliptically fibered. We will argue, on the one hand, that the strategy to identify axion-like fields is very similar to the orientifold approach and amount to a detailed study of the complex structure moduli space of X_4 . The explicit computations are, however, mathematically and technically more involved [79–83]. On the other hand, the F-theory approach is a significant generalization. In order to appreciate this, we stress that the complex structure moduli of X_4 capture not only the degrees of freedom of τ and the complex structure moduli of Y_3 , but also the seven-brane moduli. Therefore, it should allow to include the inflationary models considered in [20, 84, 85].

³A recent discussion on mirror symmetry for elliptic fibrations can be found in [58, 59].

To begin with we recall some facts about the complex structure dependent Kähler potential and flux superpotential of an F-theory compactification on X_4 . Let us denote by Ω_4 the $(4, 0)$ -form on X_4 , which is known to vary holomorphically over the complex structure moduli space with local coordinates z^I . Classically the Kähler potential for z^I takes the form

$$K^{\text{F}} = -\log \left[i \int_{X_4} \Omega_4 \wedge \bar{\Omega}_4 \right] + \dots \quad (2.13)$$

where the dots again indicate terms depending on other moduli or matter fields of the set-up. The superpotential is now induced by a real 4-form flux G_4 and takes the form

$$W^{\text{F}} = \int_{X_4} G_4 \wedge \Omega_4 + \dots \quad (2.14)$$

where the dots indicate possible non-perturbative corrections. The Kähler potential (2.13) and superpotential can be obtained by taking the M-theory to F-theory limit [29, 57]. In general, both are complicated functions of the moduli z^I , which can, however, be evaluated for certain given smooth Calabi-Yau fourfolds. Both (2.13) and (2.14) are true generalizations of the weak coupling counterparts (2.2), (2.3) not only because they depend also on the seven-brane moduli, but also because they capture information about the τ expansion, and hence the g_s corrections, beyond the leading terms present at large $\text{Im}(\tau)$. It should be stressed, however, that particularly the Kähler potential is likely to admit several perturbative and non-perturbative corrections and it remains a challenging task to examine these in detail.

As in the weak coupling setups of section 2.1 one can introduce the periods

$$\Pi^{\mathcal{I}}(z) = \int_{\Gamma_{\mathcal{I}}} \Omega_4 , \quad (2.15)$$

where $\Gamma_{\mathcal{I}}$ is a basis of four-cycles of $H_4(X_4, \mathbb{Z})$, with intersection product

$$\eta_{\mathcal{I}\mathcal{J}} = \Gamma_{\mathcal{I}} \cap \Gamma_{\mathcal{J}} . \quad (2.16)$$

In terms of these quantities one expresses (2.13) and (2.14) as

$$K = -\log \left[i \Pi^{\mathcal{I}} \eta_{\mathcal{I}\mathcal{J}} \bar{\Pi}^{\mathcal{J}} \right] + \dots , \quad W = N_{\mathcal{I}} \Pi^{\mathcal{I}} + \dots , \quad (2.17)$$

where $N^{\mathcal{I}} = \int_{\Gamma_{\mathcal{I}}} G_4$ are the flux quanta. Despite the similarities with (2.8), the proper treatment of the periods on Calabi-Yau fourfolds is significantly more involved [79–83]. This can be traced back to the fact that there is no underlying special geometry, as it is present for Calabi-Yau threefolds, that dictates already key features of the couplings.

It should be stressed that the Calabi-Yau fourfolds X_4 used in order to describe an F-theory compactification have to admit an elliptic fibration, or rather a two-torus fibration. Furthermore, it is often the case that also the mirror dual geometry Y_4 to X_4 admits an elliptic fibration. Let us assume that we have a pair X_4, Y_4 of mirror manifolds, which are both elliptically fibered. For both geometries one can then introduce the Mordell-Weil groups $\text{MW}(X_4)$ and $\text{MW}(Y_4)$. On the one hand, the rank of $\text{MW}(X_4)$ is giving the number of massless $U(1)$ gauge fields in the effective theory, and therefore a study of $\text{MW}(X_4)$ is of key importance in F-theory model building [60–78]. On the other hand, the rank of $\text{MW}(Y_4)$ has no known physical meaning in the effective theory. However, as for the Calabi-Yau threefolds, one can check that if the rank $N - 1$ of $\text{MW}(Y_4)$ is sufficiently large, then the complex structure moduli space will admit an axion at the special F-point introduced in subsection 2.2. Indeed, for $N > 3$ the monodromy around this point turns out to be of infinite order and one encounters models of *Mordell-Weil inflation*. It would be very exciting to examine thoroughly if the presence of such axion points in complex structure moduli space is generic in F-theory geometries.

3 Special points and axions in one-parameter threefolds

In this section we systematically study the special points in the complex structure moduli space of Calabi-Yau threefolds. To be explicit in our examination we restrict to Calabi-Yau manifolds X_3 with one complex structure modulus z , i.e. we consider geometries with $h^{2,1}(X_3) = 1$. As described in section 2 the $\mathcal{N} = 1$ Kähler potential and superpotential are determined by the periods of Ω . They can undergo a monodromy transformations about special points of the geometry. In one parameter models the candidates to be special points are the large complex structure point, the small complex structure point, and the conifold point. We parametrize the moduli space such that these points are located at $z = 0$, $z = \infty$, and $z = 1$, respectively, see figure 2. The aim of this section is to formulate a systematic approach to identifying the axion with a shift symmetry from the monodromy transformations present about these different special points in the complex structure moduli space. We will see that this also allows to systematically derive the inflationary potential, which turns out to be fully determined by some simple topological invariants of the manifold under study.

For the sake of concreteness we will restrict ourselves in this section to studying one-modulus Calabi-Yau threefolds X_3 constructed as mirrors of complete intersections on an ambient projective space.⁴ This mirror dual to X_3 will be denoted by Y_3 . Our

⁴We will use well known ideas in the context of mirror symmetry, so our discussion will be somewhat

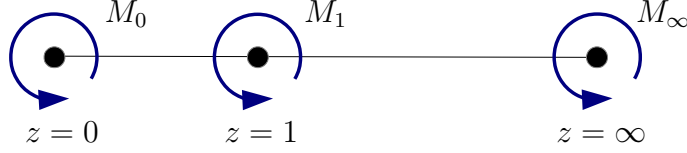


Figure 2. Schematic depiction of the one-parameter moduli space of Calabi-Yau threefolds with three special points. M_0 , M_1 , and M_∞ denote the monodromy matrices around these points.

ideas generalize easily to more involved configurations, using the techniques described in [39, 40]. It is a well known fact that periods of the holomorphic three-form Ω in such spaces satisfy a differential equation known as the Picard-Fuchs equation, which we now describe. Consider a complete intersection $Y_3 = \mathbb{P}^n[d_1 \dots d_k]$, where the notation means the complete intersection of k generic homogeneous polynomials of degrees d_1, \dots, d_k in \mathbb{P}^n . In order for Y_3 to be Calabi-Yau we need that $\sum_{i=1}^k d_i = n + 1$, and in order for it to be a threefold we need $n - k = 3$. It was proven in [40] that the periods Π on X_3 satisfy the Picard-Fuchs equation

$$\tilde{\mathcal{L}}_d \Pi(z) \equiv \left(\theta^{n+1} - \left\{ \prod_{i=1}^k (d_i \theta)(d_i \theta - 1) \cdots (d_i \theta - d_i + 1) \right\} z \right) \Pi(z) = 0 \quad (3.1)$$

with z a complex structure parameter for X_3 (with $z = 0$ corresponding to large volume in Y), and $\theta = z \partial / \partial z$. Notice that we can always factorize $\tilde{\mathcal{L}}_d = \theta^k \mathcal{L}_d$, so we can find periods by solving the simpler equation

$$\mathcal{L}_d \Pi(z) = \left(\theta^4 - z \prod_{i=1}^k d_i (d_i \theta + 1) \cdots (d_i \theta + d_i - 1) \right) \Pi(z) = 0. \quad (3.2)$$

We have commuted z to the left for later convenience. Notice that both terms in \mathcal{L}_d are fourth order in θ for Calabi-Yau threefolds. We can make this more manifest by writing

$$\mathcal{L}_\alpha \Pi(z) \equiv \left(\theta^4 - \kappa z \prod_{i=1}^4 (\theta + \alpha_i) \right) \Pi(z) = 0 \quad (3.3)$$

with $\kappa = \prod_i d_i^{d_i}$, and $\{\alpha_i\}_{i=1, \dots, k} = \{1/d_1, \dots, (d_1 - 1)/d_1, 1/d_2, \dots, (d_k - 1)/d_k\}$. It is then convenient to introduce $u = \kappa z$ and finally write

$$\mathcal{L}_\alpha \Pi(u) = \left(\theta^4 - u \prod_{i=1}^4 (\theta + \alpha_i) \right) \Pi(u) = 0. \quad (3.4)$$

concise at points. For nice reviews of the background material we refer the reader to [35–37].

In what follows we will use the u and z variables interchangeably, depending on what is most convenient.

3.1 Analysis of monodromies

Before going into technical details, let us comment on the monodromy transformations found close to the different special points in the complex structure moduli space of Calabi-Yau threefolds. Near the large complex structure point ($z = 0$) and the conifold point ($z = 1$) the monodromy transformations are unipotent matrices. By definition this means that $(T - I)^{m-1} \neq 0$ and $(T - I)^m = 0$ for some m called the index, and T being the monodromy matrix. These matrices are automatically of infinite order, i.e. there exists no n such that $T^n = I$. As we already mentioned in section 2 this fact implies the presence of a natural axion with an approximate continuous shift symmetry in the four-dimensional effective theory. More in general, around the special points we study in this paper one has a monodromy matrix satisfying $(T^p - I)^m = 0$. We want to stress here that the qualitative form of the scalar potential in the effective theory seems to be, to a large extent, determined by the properties of the monodromy matrix alone, in particular by the value of the indexes m and p , although we have not worked out the general dictionary. It would be interesting to go further in this direction and look for a direct and systematic way of extracting from the monodromy matrix the information about the low energy physics of the axion, without following the somewhat painful route of computing the periods that we take in this paper.

In particular, the solutions of (3.4) expanded near the large complex structure point have unipotent matrix of index 4, i.e. $(T[0] - I)^4 = 0$. This implies (as we will see in more detail in section 3.2) that a natural basis of periods around this point is given at leading order by $(1, \log(z), \log^2(z), \log^3(z))$, giving rise to the four-dimensional effective theory already discussed in section 2.2 which yields a polynomial-type potential for the axion. In the examples that we study the monodromy matrix near the conifold point is instead unipotent of index 2, i.e. $(T[1] - I)^2 = 0$, and the natural basis of periods is given by $(1, t_c, t_c^2, t_c \log(t_c))$ with t_c some appropriate local coordinate. The scalar potential in this case turns out to acquire a cosine-type form.

The small complex structure point is in this sense much more richer than its partners, since the properties of the monodromy matrix depend on the structure of the α_i . Let us now advance the main result that we will find: the existence of special points of infinite monodromy around $z = \infty$ will depend on the structure of the α_i : if all the α_i are different the monodromy around $z = \infty$ will be of finite order, and thus there will be no natural candidate for the axion, but if two or more α_i coincide there

will be monodromy of infinite order, and potentially interesting inflationary physics hidden deep in the complex structure moduli space of the Calabi-Yau threefold X_3 . In addition, the number of coincident α_i will determine the index of the monodromy matrix since it will determine in turn the highest power of the logarithms appearing in the expansion of the periods. For instance, if two α_i coincide we expect a very similar behavior to that of the conifold point. In contrast, if four α_i coincide the behavior will be similar to that of the large complex structure point.

This can be motivated in a general way using the formalism in [41], that applies whenever the Picard-Fuchs equation becomes of the form (3.4). Consider for example the case $\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4$, which we will study in detail momentarily. According to the results in [41], in a certain natural basis of periods (the *Jordan* basis in the language of [41]) the monodromy matrix around $z = \infty$ takes the (exponentiated normal Jordan) form

$$T[\infty] = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 & 0 & 0 \\ 0 & e^{-2\pi i \alpha_2} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \alpha_3} & 0 \\ 0 & 0 & 2\pi i e^{-2\pi i \alpha_3} & e^{-2\pi i \alpha_3} \end{pmatrix}. \quad (3.5)$$

Such a monodromy transformation arises from periods with the leading behavior at large z given by $(z^{-\alpha_1}, z^{-\alpha_2}, z^{-\alpha_3}, z^{-\alpha_3} \log(z))$. The last period is the logarithmic term that we are after. For very large values of $|z|$ the system will have an approximate shift symmetry under $z \rightarrow e^{i\beta} z$, and we can identify the approximate flat direction with the phase of z .

Similarly, if $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$ the monodromy on the Jordan basis becomes

$$T[\infty] = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 & 0 & 0 \\ 2\pi i e^{-2\pi i \alpha_1} & e^{-2\pi i \alpha_1} & 0 & 0 \\ 0 & 0 & e^{-2\pi i \alpha_3} & 0 \\ 0 & 0 & 2\pi i e^{-2\pi i \alpha_3} & e^{-2\pi i \alpha_3} \end{pmatrix} \quad (3.6)$$

and we expect two independent periods to have log behavior close to $|z| = \infty$.

In this way one can obtain the expected log behavior of a given geometry close to $z = \infty$. This gives a useful guide in searching for geometries having axions with different qualitative forms of their potential. We will now analyze in detail a number of situations where the above expectation for the behavior of the periods can be checked, and compute the low energy physics (in the complex structure sector, ignoring other aspects of the background) for the corresponding flux compactifications.

3.2 A symplectic integral basis of periods at large complex structure

Our first task is to find an integral and symplectic basis of solutions Π_i to equation (3.4). Let us start by finding a basis of solutions, not necessarily integral nor symplectic. Since we will want to do various analytic continuations, a convenient choice is to write the basis of solutions in terms of a Mellin-Barnes representation. A nice way of constructing Mellin-Barnes representations of the solutions that fits perfectly the task at hand starts by noticing [41] that (3.4) is precisely a specific type of Meijer G equation. The solutions to such an equation are aptly named Meijer G functions, and their integral representation is well known [86]:

$$U_j(z) = \frac{1}{(2\pi i)^{j+1}} \int_C ds c(s) (\Gamma(s+1)\Gamma(-s))^{j+1} (ze^{\pi i(j+1)})^s \quad (3.7)$$

with [40]

$$\begin{aligned} c(s) &\equiv \kappa^s \frac{\prod_{i=1}^4 \Gamma(s + \alpha_i)}{\Gamma(s+1)^4 \prod_{i=1}^4 \Gamma(\alpha_i)} \\ &= \frac{\prod_{i=1}^k \Gamma(d_i s + 1)}{\Gamma(s+1)^{n+1}}. \end{aligned} \quad (3.8)$$

The integration contour C in (3.7) is taken to be a straight line going from $\varepsilon - i\infty$ to $\varepsilon + i\infty$, with ε a small negative number such that $-\varepsilon < \alpha_i \forall \alpha_i$ as depicted in figure 3 by the vertical part of the (dotted or continuous) green contours. An easy exercise shows that $\mathcal{L}_\alpha U_j(z) = 0$.

Clearly \mathcal{L}_α is a linear operator, so let us try to find an integral and symplectic basis of solutions by taking linear combinations of the $U_j(z)$. The basic idea is the following [42]: the periods of Ω in an integral and symplectic basis of $H_3(X_3, \mathbb{Z})$ can be understood physically as the masses of a set of $D3$ branes⁵ wrapping a set of special Lagrangian representatives of the basis elements.⁶ By mirror symmetry, these branes will be mapped to a set of $D(2p)$ branes on Y_3 . The mass of such $D(2p)$ branes can be calculated reliably in the large volume limit. By matching with the behavior of the basis of solutions $U_j(z)$ near $z = 0$, and making use of the mirror map, we can fix an integral basis of periods.

⁵We take the convention that a Dp brane wraps a p -cycle in the Calabi-Yau.

⁶Whether such representatives actually exist at a given point in moduli space depends on considerations of stability, see [37] for a review. This subtlety is not important for the argument here, since the objects we construct will exist in the large volume regime of the mirror Y_3 , and this is enough to determine a basis of homology with the desired properties.

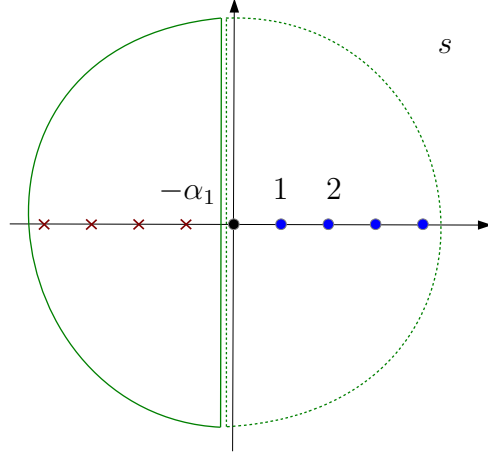


Figure 3. Schematic depiction of the s -plane including two contours, dotted green line and solid green line. The crosses indicate poles at $s = -\alpha_i - n$ with $n = \{0, 1, 2, \dots\}$, while the small spheres indicate poles at $s = \{0, 1, 2, \dots\}$.

We are interested in the expansion of (3.7) around $u = 0$ keeping only the classical terms (i.e. log and constant terms, but no positive powers of z). For computing this we can close the integral to the right, as depicted in figure 3 by a dotted green line, and compute the residue at $s = 0$. The other poles we pick up inside the contour (coming from $\Gamma(-s)$) at $s = \{1, 2, \dots\}$ will have a residue going as z^k with $k > 0$, so for the perturbative expansion they can be ignored. In this regime we have

$$U_j(z) = -\text{Res}_{s=0} \left\{ \frac{1}{(2\pi i)^j} c(s) (\Gamma(s+1)\Gamma(-s))^{j+1} (ze^{\pi i(j+1)}) \right\}. \quad (3.9)$$

The rest of the exercise is lengthy but straightforward, one just needs the expansion of the Γ function to sufficiently high order

$$\Gamma(-s) = -\left[\frac{1}{s} + \gamma + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) s + \frac{1}{6} \left(\gamma^3 + \frac{\gamma\pi^2}{2} + 2\zeta(3) \right) s^2 \right] + \mathcal{O}(s^3) \quad (3.10)$$

with γ the Euler-Mascheroni constant, a few identities relating to the digamma function $\psi(s) = \partial_s \log(\Gamma(s))$ and its derivatives with respect to s

$$\psi(1) = -\gamma, \quad \partial_s \psi(1) = \frac{\pi^2}{6}, \quad \partial_s^2 \psi(1) = -2\zeta(3), \quad (3.11)$$

and finally some identities relating $c(s)$ to topological data of Y_3 [40]

$$c(0) = 1, \quad \partial_s c(0) = 0, \quad \partial_s^2 c(0) = b\pi^2, \quad \partial_s^3 c(0) = \frac{6}{\mathcal{K}} \zeta(3) \chi, \quad (3.12)$$

where

$$\begin{aligned} \chi &= \int_{Y_3} c_3(TY_3), \quad \mathcal{K} = \int_{Y_3} D \wedge D \wedge D, \quad b = \frac{1}{3\mathcal{K}} \int_{Y_3} c_2(TY_3) \wedge D \\ \hat{b} &= \frac{\mathcal{K}}{8} b = \frac{1}{24} \int_{Y_3} c_2(TY_3) \wedge D. \end{aligned} \quad (3.13)$$

Here D is a divisor of Y_3 generating $H^2(Y_3, \mathbb{Z})$, coming from the restriction to Y_3 of the hyperplane of the ambient toric space, and $c_i(TY_3)$ are the Chern classes of the tangent bundle of Y_3 .

With all this in place, one can immediately compute the periods, for example

$$\begin{aligned} U_1(z) &= -\text{Res}_{s=0} \left\{ \frac{1}{2\pi i} c(s) (\Gamma(s+1)\Gamma(-s))^2 (ze^{2\pi i}) \right\} \\ &= -\text{Res}_{s=0} \left\{ \frac{1}{2\pi i} c(s) \Gamma(s+1)^2 \left(\frac{1}{s^2} + 2\gamma \frac{1}{s} \right) (ze^{2\pi i}) \right\} \\ &= \frac{1}{2\pi i} \left[-2\gamma \underbrace{(c(0)\Gamma(1)^2)}_1 - \partial_s (c(s)\Gamma(1+s)^2 (ze^{2\pi i s})) \Big|_{s=0} \right] \\ &= \frac{1}{2\pi i} \left[-2\gamma - (2\psi(1) + \log(ze^{2\pi i})) \right] \\ &= -\frac{\log(z)}{2\pi i} - 1. \end{aligned} \quad (3.14)$$

The other basis elements can be expanded similarly, we obtain (up to $\mathcal{O}(z)$ terms)

$$U_0(z) = 1 \quad (3.15a)$$

$$U_1(z) = -\frac{\log(z)}{2\pi i} - 1 \quad (3.15b)$$

$$U_2(z) = \frac{1}{2} \left(\frac{\log(z)}{2\pi i} \right)^2 + \frac{3}{2} \frac{\log(z)}{2\pi i} + \frac{8-b}{8} \quad (3.15c)$$

$$U_3(z) = -\frac{1}{6} \left(\frac{\log(z)}{2\pi i} \right)^3 - \left(\frac{\log(z)}{2\pi i} \right)^2 + \frac{\log(z)}{2\pi i} \frac{(3b-44)}{24} + \frac{(2b-8)}{8} - \frac{\chi}{(2\pi i)^3 \mathcal{K}} \zeta(3). \quad (3.15d)$$

The mirror map can now be constructed as usual by taking a period linear in $\log(z)$ with the right coefficient. A convenient choice, given our basis of solutions, is

$$t = -\frac{U_1(z) + U_0(z)}{U_0(z)} = \frac{1}{2\pi i} \log(z) + \mathcal{O}(z) \quad (3.16)$$

where t denotes the complexified Kähler modulus of Y_3 . More explicitly, the complexified Kähler two-form on Y_3 is given by $B + iJ = tD$, where B is the NS-NS B-field and J is the Kähler form.

Now that we have a basis of periods, and its expansion around the mirror of the large volume point of Y_3 , let us construct an integral basis by matching with the periods of a suitable basis of branes on Y_3 . A natural integral (but not necessarily symplectic) basis of objects in this regime is given by

- A $D6$ wrapping Y_3 .
- A $D4$ on the divisor D .
- A $D2$ on the curve C dual to D ($C \cdot D = 1$).
- A $D0$ on a point on Y_3 .

Notice that are underspecifying the branes: at the very least we should describe the bundle on them. Or, more precisely, we should be specifying an object in the derived category of coherent sheaves. We will do so momentarily.

We follow the notation and conventions of [37], we refer the reader to this review for the requisite background material. In the simplest case, a brane is defined by a sheaf with support on a submanifold S of Y_3 (possibly $S = Y_3$). If the inclusion map is $i: S \hookrightarrow Y_3$, such a brane is described by a sheaf $i_*\mathcal{E}$. As explained in [37], to such a brane we can assign a charge vector

$$\Gamma(i_*\mathcal{E}) = \text{ch}(i_*\mathcal{E})\sqrt{\text{Td}(TY_3)} \quad (3.17)$$

with $\text{Td}(TX)$ the Todd class of the tangent bundle to Y_3 . The expression simplifies for sheaves supported on a submanifold S , in this case [37]

$$\Gamma(i_*\mathcal{E}) = PD[S] \wedge \text{ch}(\mathcal{E}) \wedge \sqrt{\frac{\text{Td}(TS)}{\text{Td}(NS)}} \quad (3.18)$$

where $PD[S]$ is the Poincaré dual form to S .

We are now in a position to identify our integral basis (not yet symplectic). The $D6$ will be the brane associated to the trivial sheaf on Y_3 , i.e. \mathcal{O}_{Y_3} . The $D0$ will be the skyscraper sheaf on a point. The $D2$ will be $j_*\mathcal{O}$ with $j: C \hookrightarrow Y_3$, and the $D4$ will be $k_*\mathcal{O}$, with $k: D \hookrightarrow Y_3$.

In the classical limit (i.e. close to $z = 0$) the periods for such a brane wrapped on a manifold S are easily computed [37]

$$Z(i_*\mathcal{O}_S) = \int_{Y_3} e^{-tD} \wedge \Gamma(i_*\mathcal{O}_S) = \int_S i^*(e^{-tD}) \wedge \sqrt{\frac{\text{Td}(TS)}{\text{Td}(NS)}} \quad (3.19)$$

with t the mirror parameter. Using well known expression for the Todd class of a bundle E

$$\text{Td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \dots \quad (3.20)$$

where we have omitted terms vanishing for complex threefolds. We get the following central charges (3.19) for our integral basis of branes (here $i_*\mathcal{O}_p$ is the trivial sheaf supported on the Dp brane)

$$Z(i_*\mathcal{O}_0) = 1 \quad (3.21a)$$

$$Z(i_*\mathcal{O}_2) = -t + 1 - g_C \quad (3.21b)$$

$$Z(i_*\mathcal{O}_4) = \frac{\mathcal{K}}{2}t^2 - \frac{1}{2}\mathcal{K}t + \frac{\mathcal{K}}{6} + \frac{1}{24} \int_D c_2(TD) \quad (3.21c)$$

$$Z(i_*\mathcal{O}_6) = -\frac{\mathcal{K}}{6}t^3 - t\hat{b} - \hat{c}, \quad (3.21d)$$

where \hat{b} and \mathcal{K} have been introduced in (3.13) and we have encoded in \hat{c} the one-loop term $\hat{c} = \frac{\zeta(3)\chi}{(2\pi i)^3}$.⁷ The integer g_C is the genus of C .

We see that the qualitative form of this set of central charges matches nicely with the structure of the periods in X_3 near $z = 0$ found in (3.15). This basis is clearly made of integral objects, but it is not necessarily symplectic. In order to achieve this, we will take integral linear combinations of these elements in such a way that the chiral spectrum in the new basis is of the symplectic form. By the Hirzebruch-Riemann-Roch theorem and the expression (3.17) the net chiral spectrum between branes A and B is counted by [37]

$$\langle A, B \rangle \equiv \int_{Y_3} \text{ch}(i_*\mathcal{E}_A^\vee) \text{ch}(i_*\mathcal{E}_B) \text{Td}(TY_3) = \int_{Y_3} \Gamma(A)^\vee \wedge \Gamma(B) \quad (3.22)$$

with $\Gamma(A)^\vee$ means that the even components of the polyform $\Gamma(A)$ should be changed sign. (Notice that this product, being mirror to an intersection product between $D3$

⁷Let us point out that the expression (3.17) should actually use the so-called Γ -class instead of the Todd class [47–49, 87–90]. The correction is precisely the \hat{c} term which we have simply added by hand to (3.21d), the rest of the central charges (3.21a)–(3.21c) are unaffected.

branes, is antisymmetric as it should, and it is always an integer.) We obtain:

$$\begin{aligned}
\langle D6, D0 \rangle &= 1 , & \langle D6, D2 \rangle &= 1 - g_C \\
\langle D6, D4 \rangle &= \frac{\mathcal{K}}{6} + \frac{1}{24} \int_D c_2(TD) + \hat{b} , & \langle D4, D2 \rangle &= -1 , \\
\langle D4, D0 \rangle &= 0 , & \langle D2, D0 \rangle &= 0 .
\end{aligned} \tag{3.23}$$

We see that the basis is almost symplectic. We can make it completely symplectic by choosing as a basis $\mathcal{S} = \{D0, -D2 + (1 - g_C)D0, -D4 + \langle D6, D4 \rangle D0, -D6\}$. The classical periods in this basis, which is now the integral symplectic basis that we are after, are

$$\Pi^{\mathcal{S}} = \begin{pmatrix} 1 \\ t \\ -\frac{\mathcal{K}}{2}t^2 + \frac{\mathcal{K}}{2}t + \hat{b} \\ \frac{\mathcal{K}}{6}t^3 + t\hat{b} + \hat{c} \end{pmatrix} \tag{3.24}$$

This agrees with the expression in [91].

We can now match this perturbative behavior with the expansion of the $U_i(z)$ basis found in (3.15), with the result that we can write the expression for the periods in the integral symplectic basis in terms of the exact solutions $U_i(z)$. We have that $\Pi_i^{\mathcal{S}} = \Xi_{ij} U_j(z)$, with

$$\Xi_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -\mathcal{K} & -2\mathcal{K} & -\mathcal{K} & 0 \\ -(\frac{\mathcal{K}b}{4} + \frac{\mathcal{K}}{6}) & -(\frac{\mathcal{K}b}{4} + \frac{7\mathcal{K}}{6}) & -2\mathcal{K} & -\mathcal{K} \end{pmatrix}. \tag{3.25}$$

This is all the information we need in order to start exploring the physics away from $z = 0$. But before turning to this problem let us comment about the four-dimensional $\mathcal{N} = 1$ effective theory that we get near the large complex structure point. Inserting the periods of (3.24) into (2.7) and (2.8) we get the following Kähler potential and the $\mathcal{N} = 1$ superpotential

$$K_{\text{lcs}} = -\log [i(\tau - \bar{\tau})] - \log \left[i \left(-\frac{1}{6} \mathcal{K}(t - \bar{t})^3 + 2\hat{c} \right) + \dots \right], \tag{3.26}$$

$$W_{\text{lcs}} = \frac{1}{6} \mathcal{N}_4 \mathcal{K} t^3 - \frac{1}{2} \mathcal{N}_3 \mathcal{K} t^2 + \left(\mathcal{N}_4 \hat{b} + \frac{1}{2} \mathcal{N}_3 \mathcal{K} + \mathcal{N}_2 \right) t + \left(\mathcal{N}_1 - \mathcal{N}_4 \hat{c} + \mathcal{N}_3 \hat{b} \right),$$

where \mathcal{N}_i was defined in (2.8) and correspond to the flux quanta. We can see that K preserves an approximate continuous shift symmetry for the real part of the field t , which would be broken by non-perturbative corrections to K as we move away from the special point $z = 0$ ($\text{Im}(t) \rightarrow \infty$). These non-perturbative contributions of $\mathcal{O}(e^{2\pi i t})$ come from the $\mathcal{O}(z)$ terms in the expansion of the periods in (3.15a)-(3.15d), and break the continuous symmetry to a discrete periodicity. This discrete symmetry remains exact since it belongs to the discrete monodromy symmetries preserved by K near the large complex structure point. However if we are close enough to the special point these corrections are negligible and we recover an approximate continuous shift symmetry for the field. This fact has been used in several occasions to realize large field inflation models within the complex structure moduli space of Calabi-Yau manifolds [19, 20, 30–32]. In these models the shift symmetry is broken by the flux induced superpotential derived above, giving rise to the usual polynomial-type potentials.

However it is particularly simple, and physically relevant, to explore the behavior of the periods close to $z = \infty$, where new types of scalar potentials arise. We turn to this problem next.

3.3 Continuation to $z = \infty$: generalities

We just obtained the expansion around $z = 0$ of the periods (3.7) by closing the integration contour in figure 3 to the right. This does not modify the value of the integral when $|u| < 1$, and in this regime the sum over the residues gives rise to a convergent series. Neither of these statements is true when $|u| > 1$, but a similar idea applies: when $|u| > 1$ one can close the contour *to the left* without modifying the integral, and the resulting sum over residues gives a convergent series. This contour is depicted in figure 3 by a solid green line,

The singular behavior of the integrand comes, as before, from the poles of the Γ functions at non-positive integral values. Recall from (3.7) that the integrand for U_j has a expression of the form (focusing on the Γ function part)

$$U_j(z) \sim \int ds \frac{\prod_{i=1}^4 \Gamma(s + \alpha_i)}{\Gamma(s + 1)^{3-j}} \dots \quad (3.27)$$

It is clear that the poles for $\text{Re}(s) < 0$ come only from the numerator, when $s = -\alpha_i - n$, with $n \in \{0, 1, \dots\}$.

Furthermore, now we see clearly the origin of our claim before that the qualitative behavior of the low energy physics close to $z = \infty$ will depend on the structure of the α_i : if they all different all the poles picked up by the integral will be single poles, and there

are no $\log(z)$ factors in the periods. If they are double poles, the corresponding terms will be linear in logs, etc. For having interesting monodromies we will want to have at least two coincident α_i . We will now analyze in detail some interesting scenarios.

We focus on the case of having double poles. Triple and quadrupole poles (corresponding to $\alpha_1 = \alpha_2 = \alpha_3 \neq \alpha_4$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$) are analyzed similarly. The particular case of all α_i equal is conceptually interesting: in this case one would expect the behavior close to $z = 0$ to be qualitatively similar to that at $z = \infty$. Study of concrete examples supports this expectation: the canonical example of this situation, $\mathbb{P}^7[2, 2, 2, 2]$ has a $z \leftrightarrow 1/z$ symmetry in moduli space, as pointed out in [38].

3.4 $\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4$

Let us consider the minimal non-trivial case in which two eigenvalues coincide,

$$\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4. \quad (3.28)$$

In this case we have two contributions to the periods coming from the single poles ($s = -\alpha_1$ and $s = -\alpha_2$) and from the double pole ($s = -\alpha_3$),

$$U_j(z) = U_j^s(z) + U_j^d(z) \quad (3.29)$$

For instance, the contribution from the single poles at $s = -\alpha_1$ and $s = -\alpha_2$ is given by ($k = 1, 2$)

$$U_j^s(z) = \text{Res} \left[\frac{1}{(2\pi i)^j} c(s) (\Gamma(s+1)\Gamma(-s))^{j+1} (ze^{\pi i(j+1)})^s \right]_{s=-\alpha_k} \quad (3.30)$$

leading to

$$U_j^s(z) = \frac{A_k}{(2\pi i)^j} \left(\frac{\sin(\pi\alpha_k)}{\pi} \right)^{3-j} e^{-\pi i(j+1)\alpha_k} (\kappa z)^{-\alpha_k} \quad (3.31)$$

with

$$A_k = \frac{\Gamma(\alpha_k)^4 \prod_{i \neq k} \Gamma(\alpha_i - \alpha_k)}{\prod_{i=1}^4 \Gamma(\alpha_i)}. \quad (3.32)$$

To compute the residue we have used the reflection formula

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}. \quad (3.33)$$

The contribution from the double pole at $s = -\alpha_3$ is given by

$$U_j^d(z) = \text{Res} \left[\frac{1}{(2\pi i)^j} \frac{\Gamma(s+\alpha_1)\Gamma(s+\alpha_2)}{\prod_{i=1}^4 \Gamma(\alpha_i)\Gamma(s+1)^4} \left(\frac{-2\gamma}{s+\alpha_3} + \frac{1}{(s+\alpha_3)^2} \right) \right. \\ \left. \times (\Gamma(s+1)\Gamma(-s))^{j+1} (ze^{\pi i(j+1)})^s \right]_{s=-\alpha_3}, \quad (3.34)$$

which yields

$$U_j^d(z) = \frac{A_3}{(2\pi i)^j} \left(\frac{\sin(\pi\alpha_3)}{\pi} \right)^{3-j} e^{-\pi i(j+1)\alpha_3} (\kappa z)^{-\alpha_3} \times (B - (3-j)\pi \cot(\pi\alpha_3) + (j+1)i\pi + \log(\kappa z)) \quad (3.35)$$

with

$$B = \psi(\alpha_1 - \alpha_3) + \psi(\alpha_2 - \alpha_3) - 4\psi(\alpha_3) - 2\gamma, \quad (3.36)$$

and A_3 as in (3.32). Here we have also used the identity

$$\psi(1-s) - \psi(s) = \pi \cot(\pi s). \quad (3.37)$$

Notice that unlike the single poles, the double pole induces a logarithmic term in the periods.

We can now write the leading expansion of the integral symplectic basis using the relation (3.25). For concreteness let us consider a particular example, given by $\mathbb{P}^5[2, 4]$. In this geometry we have $\mathcal{K} = 8$, $\hat{b} = b = \frac{7}{3}$, $\chi = -176$, $\kappa = 2^{10}$, and $\alpha_i = \{\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}\}$. The periods in the Meijer basis read

$$U_j(z) = U_j^s(z) + U_j^d(z) \quad (3.38)$$

where

$$U_j^s(z) = \frac{1}{(2\pi i)^j} \frac{\Gamma(1/4)^6}{(2\pi^3)^{1/2}} (\sqrt{2}\pi)^{j-3} e^{-\pi i(j+1)/4} (\kappa z)^{-1/4} + \mathcal{O}(z^{-3/4}), \quad (3.39)$$

$$U_j^d(z) = -\frac{1}{(2\pi i)^j} \frac{4\pi}{\pi^{3-j}} e^{-\pi i(j+1)/2} (\kappa z)^{-1/2} (2\log(2) + 4 + (j+1)i\pi + \log(\kappa z)) + \mathcal{O}(z^{-3/4}). \quad (3.40)$$

To compute the second contribution we have evaluated the Gamma functions at the particular values of α_i corresponding to $\mathbb{P}^5[2, 4]$, obtaining

$$A_3 = \frac{\Gamma(1/2)^2 \Gamma(-1/4) \Gamma(1/4)}{\Gamma(1/4) \Gamma(3/4)} = -4\pi \quad (3.41)$$

$$B = \psi(-1/4) + \psi(1/4) - 4\psi(1/2) - 2\gamma = 2\log(2) + 4. \quad (3.42)$$

Using the change of basis (3.25) this gives rise to the following expansion of the integral symplectic periods around the small complex structure point

$$\Pi^S = \frac{4i}{\pi^2} u^{-1/2} \begin{pmatrix} B + i\pi + \log(u) \\ \frac{1}{2}(-B - \log(u)) \\ \frac{1}{4}(-B\mathcal{K} - 8\pi - \mathcal{K}\log(u)) \\ -\frac{1}{24}(24\hat{b} - \mathcal{K})(B + \log(u)) \end{pmatrix} + \frac{\Gamma(1/4)^6}{4\pi^4 \sqrt{\pi}} u^{-1/4} \begin{pmatrix} -e^{3\pi i/4} \\ \frac{i}{\sqrt{2}} \\ \frac{1}{2}\mathcal{K}e^{i\pi/4} \\ \frac{i}{3\sqrt{2}}(6\hat{b} - \mathcal{K}) \end{pmatrix} + \dots \quad (3.43)$$

(recall $u \equiv \kappa z$) which implies a Kähler potential given by

$$\begin{aligned} K &= -\log(i\Pi^{S\dagger}\Sigma\Pi^S) \\ &= -\log\left(\frac{1}{48\pi^6|u|}(\pi^3 2\operatorname{Re}(a_1 \log(u)) - \pi^2 2\operatorname{Re}(a_2 u^{1/4}) + a_3 |u|^{1/2} + a_4) + \dots\right) \end{aligned} \quad (3.44)$$

where

$$a_1 = 32(24i + 24\hat{b} - \mathcal{K}) = 1536 - 768i \quad (3.45)$$

$$a_2 = A_1(2 + 2i)(4 - 4i)(6i + 6b - \mathcal{K}) = A_1(96 + 96i) \quad (3.46)$$

$$a_3 = A_1^2(12\hat{b} - 5\mathcal{K}) = -12A_1 \quad (3.47)$$

$$a_4 = 64B(24b - \mathcal{K}) = 3072B \quad (3.48)$$

and $A_1 = \frac{\Gamma(1/4)^6}{(2\pi^3)^{1/2}}$. Notice that the leading term when $z \rightarrow \infty$ is of order $\mathcal{O}(|u|^{-1}\log(u))$ while the corrections go as $|u|^{-1}u^{\alpha_i}$. The cross term $\mathcal{O}(u^{1/4}\log(u))$ vanishes since the coefficient is proportional to $24\hat{b} - 7\mathcal{K} = 0$. Here the coefficients a_i are complex combinations of the topological intersection numbers so in general will be complex, implying a quadratic dependence of e^{-K} on the phase of u (unlike the sinusoidal modulations coming from the polynomial corrections $\mathcal{O}(u^{\alpha_i})$). However this term is suppressed by $|u|^{-1}$ so can be neglected if we are close enough to the special point (as well as the subleading corrections that are not proportional to $\log(z)$) such that we recover an approximate shift symmetry near the special point $u = \infty$.

All the integral symplectic periods show a behavior on the coordinate u given by $\mathcal{O}(u^{-1/2}(\log(u) + \dots)) + \mathcal{O}(u^{-1/4})$. Hence schematically the superpotential will be given by

$$W = N_{eff}u^{-1/2}(\log(u) + \dots) + N'_{eff}u^{-1/4} + \dots \quad (3.49)$$

with N_{eff}, N'_{eff} some effective coefficients depending also on the NSNS and RR fluxes. This implies a scalar potential of the following form

$$V = V_0(|u|) + f(|u|)\cos(\alpha_3 - \alpha_1)\phi + \dots \quad (3.50)$$

with ϕ (the phase of u) being the natural candidate for the axion. Notice that $V_0(|u|)$ and $f(|u|)$ are functions only on the modulus $|u|$ so the leading term in the axion is a cosine term. This behaviour will also appear near the conifold point as we will see in section 3.6. The difference is that here the effective decay constant also depends on the structure of the α_i and there might be an enhancement given by $(\alpha_3 - \alpha_1)^{-1}$ (a factor 4 in the case of $\mathbb{P}^5[2, 4]$).

Let us come back momentarily to the general discussion in section 3.1, and check that the generic expectations in that section are fulfilled in this example. Clearly, taking appropriate linear combinations of (3.39) one can construct a basis of periods with the leading behavior $(z^{-1/4}, z^{-3/4}, z^{-1/2}, \log(z)z^{-1/2})$. The monodromy behavior of this basis under $z \rightarrow e^{2\pi i}z$ is given by (3.5). Taking the monodromy in this Jordan basis, and changing back to our basis (3.7), we obtain a monodromy matrix for $\mathbb{P}^5[2, 4]$ given by

$$T^U[\infty] = \begin{pmatrix} 1 & 6 & 8 & 8 \\ -1 & -5 & -8 & -8 \\ 1 & 5 & 9 & 8 \\ -1 & -5 & -9 & -7 \end{pmatrix} \quad (3.51)$$

which in the integral symplectic basis becomes

$$T^S[\infty] = \begin{pmatrix} -5 & -8 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 6 & 8 & -1 & 1 \end{pmatrix}. \quad (3.52)$$

It is easy to check that $(T[\infty]^4 - I)^2 = 0$.

3.5 $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$

Let us consider the case in which $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$. The poles for $\Re(s) < 0$ are located at $s = -\alpha_1 - n$ and $s = -\alpha_3 - n$ with $n \in \{0, 1, \dots\}$. Unlike the previous case, now all the poles are double. Integration around these poles gives a contribution of the exact form (3.35) and another also of the form (3.35) with $\alpha_3 \rightarrow \alpha_1$.

One example manifold with these properties is $\mathbb{P}^5[3, 3]$, with $\alpha_i = \{\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\}$, $\mathcal{K} = 9$, $b = 2$, $\hat{b} = \frac{9}{4}$, $\kappa = 3^6$ and $\chi = -144$. The parameters A_k, B become

$$\begin{aligned} A_1 &= \frac{\Gamma(1/3)^2 \Gamma(1/3)^{2^2}}{\Gamma(2/3)} = \frac{3}{4\pi^2} \Gamma(1/3)^6, & A_2 &= \frac{\Gamma(2/3)^2 \Gamma(-1/3)^{2^2}}{\Gamma(1/3)} = \frac{12\pi^2}{A_1}, \\ B_1 &= -2\psi(1/3) - 2\gamma, & B_2 &= B_1 + 6 - \frac{2\pi}{\sqrt{3}}, \end{aligned} \quad (3.53)$$

and the Kähler potential at leading order is given by

$$\begin{aligned} K = -\log \left(-\frac{9}{128A_1^2\pi^6|u|^{4/3}} \left(A_1^4 6\sqrt{3}(1-i\sqrt{3})|u^{1/3}\log(u)|^2 + \right. \right. \\ \left. \left. + 72\sqrt{3}\pi^2 A_1 \log(u)|^2((u^{1/3} - \bar{u}^{1/3}) - i\sqrt{3}(u^{1/3} + \bar{u}^{1/3})) \right) + \dots \right), \end{aligned} \quad (3.54)$$

Note that the leading term on the Kähler potential goes as $\mathcal{O}(|u|^{-2/3} \log |u|)$ so we again recover an approximate shift symmetry for the phase of u in the small complex structure limit $u \rightarrow \infty$. This shift symmetry would be broken by a flux-induced superpotential given by

$$W = N_{eff} u^{-1/3} (\log(u) + \dots) + N'_{eff} u^{-2/3} (\log(u) + \dots) + \dots \quad (3.55)$$

In conclusion, when we have one or two pairs of coincident roots α_i , the Kähler potential near $u = \infty$ goes as

$$K = -\log(\mathcal{O}(|u^{-\alpha_1}|^2 \log |u|)) \quad (3.56)$$

where α_1 would be the lowest repeated root. In this limit we recover a shift symmetry for the phase of u which is perturbatively broken by a superpotential taking the form

$$W = \sum_i N_{eff_i} u^{-\alpha_i} (\log(u) + \dots) + \dots \quad (3.57)$$

with the index i running over the double repeated roots.

3.6 The conifold point

All the equations of the form (3.4) have a conifold-like singularity at $u = 1$ around which the monodromy matrix satisfies $(T[1] - 1)^2 = 0$. Unfortunately the Picard-Fuchs differential equations around this point do not take the form of a Meijer G equation, so the method followed in the previous cases does not apply. In [92] it was proven that (for the manifolds under study here) there is always a basis of periods in which the monodromy matrix around the conifold point takes the form

$$T[1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\mathcal{K} & 0 & 1 \end{pmatrix} \quad (3.58)$$

This basis is based on three power series solutions and one logarithmic solution for the Picard-Fuchs equation. While the expansion of the periods has been extensively studied for the case of the quintic hypersurface in \mathbb{P}^4 [93], there is no such study for general one-parameter Calabi-Yau threefolds. It is beyond the scope of this paper to give the quantitative and general behavior for the periods in any one-parameter threefold, so we will restrict our analysis to the quintic. However since the qualitative form of

the periods is general for any one-parameter Calabi-Yau manifold with Picard-Fuchs equation given by (3.4), we expect similar behavior in other manifolds in the class we study.

Let us consider then the mirror of the quintic manifold $\mathbb{P}^4[5]$. An integral symplectic basis of periods which undergo the monodromy given by (3.58) when circling around the conifold is given by [91, 93]

$$\Pi = \omega_0(z) \begin{pmatrix} 1 \\ t_c \\ \frac{\mathcal{K}}{2} \frac{1}{2\pi i} t_c^2 \\ -\frac{\mathcal{K}}{2\pi i} t_c \log t_c - \frac{\mathcal{K}}{2} \frac{1}{2\pi i} t_c \end{pmatrix}. \quad (3.59)$$

Recall that $\mathcal{K} = 5$ for the quintic. The coordinate t_c can be written in terms of the u coordinate appearing in (3.4) as

$$t_c = \delta + \dots, \text{ with } \delta = \frac{u-1}{u} \quad (3.60)$$

Hence the conifold point is located at $\delta = 0$ and circling a complete period around this point is equivalent to transform $t_c \rightarrow t_c e^{2\pi i}$. Using the above formula we get the following Kähler potential

$$K = -\log \left[\frac{-\mathcal{K}}{\pi} |t_c|^2 \log |t_c| + \frac{1}{2} (t_c - \bar{t}_c)^2 + \dots \right] \quad (3.61)$$

The Kähler potential is invariant under the discrete transformation

$$\theta \rightarrow \theta + 1 \quad (3.62)$$

where θ is the phase of t_c , ie. $t_c = |t_c| e^{2\pi i \theta}$. This exact discrete symmetry is inherited from the monodromy transformation which leaves the symplectic form invariant, and can be promoted to an approximate shift symmetry if we are close enough to the special point.

The addition of fluxes induces a superpotential given by

$$W = -\mathcal{N}_4 \mathcal{K} t_c \log(t_c) + \frac{1}{2} \mathcal{N}_3 \mathcal{K} t_c^2 + \left(\mathcal{N}_2 - \frac{1}{2} \mathcal{N}_4 \mathcal{K} \right) t_c + \mathcal{N}_1 \quad (3.63)$$

where \mathcal{N}_i denote the flux quanta introduced in (2.8). By using the supergravity formulae we obtain the following scalar potential,

$$V = V_0(|t_c|) - \mathcal{N}_2 \mathcal{N}_3 \frac{\pi}{\mathcal{K}} (\log |t_c|^2) |t_c| \cos(\theta) + \dots \quad (3.64)$$

which corresponds to a cosine-type potential for the axion θ (phase of t_c). While a detailed analysis of the scalar potential is beyond the scope of this paper, we can already remark that this point of the moduli space could be a good candidate to construct a model of natural inflation.

4 Elliptic fibrations and Mordell-Weil inflation

We now study a simple two-parameter setup in which inflation can appear beyond the large complex structure point, namely mirrors for genus one fibrations.⁸ We will focus on the large base limit, which effectively turns the problem into a one-dimensional problem, and show how infinite order monodromies can arise close to the small fiber limit. This observation was already made in [59], and some of the relevant technical details were presented there. Technically the analysis is fairly similar to the one presented in the last section, so we will content ourselves with outlining some of the most relevant points of the system, without going into much detail.

A T^2 can be realized in a number of different ways. The most conventional is as a degree 6 hypersurface on a \mathbb{P}^{231} , but other realizations are as a $\mathbb{P}^{112}[4]$ hypersurface, a $\mathbb{P}^2[3]$ hypersurface or a $\mathbb{P}^3[2, 2]$ complete intersection. Other realizations exist, see for example [66, 72] for recent discussions, but we will focus on the ones just presented. They are commonly known in the literature as E_8 , E_7 , E_6 and D_5 models, respectively.

Using the results in [40] one quickly obtains that the Picard-Fuchs equation for the T^2 in these realizations is given by

$$\mathcal{L} = \theta^2 - z(\theta + \alpha_1)(\theta + \alpha_2) \quad (4.1)$$

with θ the logarithmic derivative on the fiber coordinate z , α_i depending on the type of the fiber realization. Concretely, for E_8 we have $\alpha_i = \{\frac{1}{6}, \frac{5}{6}\}$, for E_7 we have $\{\frac{1}{4}, \frac{3}{4}\}$, for E_6 $\{\frac{1}{3}, \frac{2}{3}\}$ and finally for D_5 one has $\{\frac{1}{2}, \frac{1}{2}\}$.

The expansion of the periods around large complex structure (i.e. small z) will have a period going as a regular power series in z and another going as $\log(z)$. Interestingly, we see a rather marked difference when we analytically continue to large z , using the techniques reviewed in section 3: for the $E_{\{8,7,6\}}$ realizations of the fiber the expansion around $z = \infty$ will be regular (all the poles are simple poles, since $\alpha_1 \neq \alpha_2$), but for

⁸These are fibrations where the fiber has the topology of a T^2 . Typically in string theory model building these appear as elliptic fibrations, which implies the existence of a section, although this condition can be relaxed (which can be interesting for model building purposes [69, 70, 75–78]) and the results here still apply.

D_5 we will have a logarithmic solution, since $\alpha_1 = \alpha_2$, and the poles will be double. Equivalently, the monodromy matrices around $z = \infty$ will be of finite order ($T[\infty]^k = I$ for some positive integer k) for $E_{\{8,7,6\}}$, but will be of infinite order for D_5 . An explicit computation of periods confirms this [59]: if we take $a = \{1, 2, 3, 4\}$ for E_8 , E_7 , E_6 or D_5 respectively, we have that the monodromy of the periods around $z = \infty$ is given in a certain natural basis by

$$T[\infty] = \begin{pmatrix} 1 - a & -1 \\ a & 1 \end{pmatrix} \quad (4.2)$$

which in a Jordan basis (recall the discussion in section 3.1) is

$$T[\infty] = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 0 & e^{-2\pi i \alpha_2} \end{pmatrix} \quad (4.3)$$

for $\alpha_1 \neq \alpha_2$, and

$$T[\infty] = \begin{pmatrix} e^{-2\pi i \alpha_1} & 0 \\ 2\pi i e^{-2\pi i \alpha_1} & e^{-2\pi i \alpha_1} \end{pmatrix} \quad (4.4)$$

if $\alpha_1 = \alpha_2$.

Let us now fiber this over a base, so we obtain a Calabi-Yau threefold. For concreteness, let us choose the base to be \mathbb{P}^2 , and the fibration to have global sections. The fibration structure in this case was given in [94]. The Picard-Fuchs equations for the whole system are given by [58, 59]

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 3\theta_2) - z_1(\theta_1 + \alpha_1)(\theta_1 + \alpha_2) \\ \mathcal{L}_2 &= \theta_2^3 - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2). \end{aligned} \quad (4.5)$$

Here z_1 is the fiber coordinate, z_2 the base, and α_i parameterize the type of the fiber, just as before. In this context the parameter a in (4.2) has a natural interpretation as the number of sections of the compactification [94].

The limit of large volume of the base corresponds formally to setting $\theta_2 = 0 = z_2$. Then the Picard-Fuchs operator (4.5) reduces to the Picard-Fuchs operator on the torus fiber (4.1), and the conclusions that we obtained there carry over straightforwardly. In particular, there is a marked difference between the $E_{\{8,7,6\}}$ fibration types and the D_5 fibration type, or more intrinsically a marked physical difference depending on the number of sections the fibration possesses.

As advanced in section 2.2, the number of sections enters in a rather interesting way in a number of important physical features of string compactifications. Here we just saw a new interesting feature of the Mordell-Weil group of the mirror of a given

compactification (assuming that the mirror is elliptically fibered): if its rank is high enough, we have a natural candidate for the axion in the limit where the fiber of the mirror is very small.

5 Conclusions

In this paper we initiated a systematic study of the global structure of the complex structure moduli space of Calabi-Yau threefolds and fourfolds with application to building inflationary models in $\mathcal{N} = 1$ flux compactifications. We focused our analysis on the Kähler potential $K_{\text{cs}} = -\log \int \Omega \wedge \bar{\Omega}$ and flux superpotential $W_{\text{cs}} = \int \Omega \wedge G$, where G is a three-form or four-form flux. Globally K_{cs} , or rather the corresponding Kähler metric, has a discrete symmetry group known as the monodromy group G_{mon} . Specific monodromies arise when encircling certain special points in the complex structure moduli space. We showed that close to some of these points, namely the ones with infinite-order monodromies, an approximate continuous symmetry in the Kähler potential K_{cs} emerges. The corresponding degree of freedom in the low-energy effective action is an axion-like complex structure modulus with an approximate shift-symmetry in the Kähler potential. Both the discrete and continuous symmetry are broken by a sufficiently general flux superpotential. Expanding W_{cs} around the points with an axion one can then evaluate the scalar potential. We have summarized our findings about the different Kähler potentials and superpotentials in section 2.2.

Let us comment on some interesting relations of our set-up to other inflationary models. Firstly, we pointed out that close to conifold points in the complex structure moduli space of Calabi-Yau threefolds there is a natural axionic degree of freedom. The continuous symmetry is then broken to a discrete symmetry by a flux superpotential and a scalar potential of cosine-type appears. It is well-known [52–56] that in some cases there is a dual description of this set-up obtained by performing a conifold transition to a resolved configuration in which the singularity is replaced by a \mathbb{P}^1 . In this dual setting, the axionic degree of freedom is a scalar that complexifies the resolution volume of the conifold singularity. The three-form flux translates after transition into the fact that there is a stack of $D5$ -branes on the resolving \mathbb{P}^1 with a superpotential from a gaugino condensate. In our orientifold set-up an $\mathcal{N} = 1$ version of this duality could be applicable. In other words, using conifold singularities and three-form fluxes one can aim to engineer aligned axion models or N-flation models.

We also briefly considered F-theory compactifications on Calabi-Yau fourfolds. These are manifestly $\mathcal{N} = 1$ configurations and the dualities arising at different points

in the complex structure moduli space have been much less studied. One interesting recent observation has been made for geometric transitions with conifold curves in the fourfold [76, 95]. Performing such a transition and identifying a complex structure modulus $z \rightarrow e^{iG}$, where G is a linear combination of R-R and NS-NS two-form degrees of freedom, one should be able to obtain the inflationary scenarios suggested in [12, 96]. It should be interesting to pursue this further.

Let us stress that despite the absence of a detailed understanding of $\mathcal{N} = 1$ dualities, the period computations can be performed using the techniques developed in [79–83], and the Kähler potential and superpotential can be computed at various points in the complex structure moduli space. It would be interesting to explicitly do that for elliptically fibered fourfolds. Since these computations allow to determine subleading corrections and can be evaluated for various flux choices, we are confident that they will allow to perform a solid analysis of tunneling probabilities between various flux branches.

It is important to end with a cautionary remark on the actual realization of an inflationary model in the proposed way. In order to exploit the fact that there is an axion at a special point in moduli space crucially requires to stabilize all moduli such that the effective action is actually evaluated near this point with the axion being the lightest field. This is a notoriously hard problem, since often all fields start to mix in both the Kähler potential and superpotential. In the Kähler potential, for instance, the complex structure moduli mix with brane moduli and can correct the $\mathcal{N} = 1$ coordinates for the cycle volumes [57, 97]. The actual field with an approximate shift-symmetry in the Kähler potential will thus eventually depend on all mixings. Furthermore, the prefactors of instanton superpotentials used to stabilize Kähler volume moduli will generically depend on the complex structure moduli. This can ruin flatness, or at least make a separate consideration of complex structure and Kähler volume moduli questionable. Axions might be useful to identify a candidate inflaton, but there is a good chance that it is not harder to build an inflationary model by simply looking for a flat direction in a scalar potential derived from string theory. In the end the hardest task is to determine in a robust way the effective scalar potentials that arise in string theory.

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